JOURNAL OF DIFFERENTIAL EQUATIONS **100**, 355–378 (1992)

Asymptotic Behavior of Strongly Monotone Time-Periodic Dynamical Processes with Symmetry

PETER TAKÁČ*

Mathematics Department, Vanderbilt University, Nashville, Tennessee 37235 and Mathematics and Computer Science Division†, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439

Received October 15, 1990

Given a strongly monotone discrete-time dynamical system $\{T^n: X \to X: n \in \mathbf{Z}_+\}$ in an open and order-convex subset X of a separable strongly ordered Banach space V, and a compact connected metrizable group G whose action on X is monotone and commutes with T, we prove under some reasonable additional hypotheses that the ω-limit set $\omega(x)$ of every stable point $x \in X$ consists of symmetric points only, i.e., $g \cdot w = w$ for all $w \in \omega(x)$ and $g \in G$. Moreover, the set of all unstable points is contained in a union of at most countably many Lipschitz manifolds of codimension one in V where each manifold is invariant under T and the action of G. This result is applied to the time-periodic, spatially independent, irreducible cooperative system of n reaction–diffusion equations

$$\frac{\partial u}{\partial t} = D(t)\,\Delta u + F(t, u) \qquad \text{for} \quad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^N$$

with spatially periodic boundary conditions in $\mathbf{R}^N$, and with an initial distribution u_0 which is continuous and satisfies the boundary conditions. If u_0 is stable, then $\omega(u_0)$ contains spatially constant functions only, and the dynamics on $\omega(u_0)$ is given by the irreducible cooperative system of n ODEs

$$\frac{du}{dt} = F(t, u) \qquad \text{for} \quad t \in \mathbf{R}^1_+.$$

Here T is the Poincaré map. If $n = 1$ then $\omega(u_0)$ is a single fixed point of T; if $n = 2$ then $\omega(v_0)$ is a single fixed point of T for each $v_0 \in \omega(u_0)$. © 1992 Academic Press, Inc.

0. INTRODUCTION

This article is motivated by the study of large-time asymptotic behavior of solutions to the following reaction–diffusion-type initial value problem

* Supported in part by the National Science Foundation under Grant DMS-8802646.
† On leave during the academic year 1990–1991.

0022-0396/92 $5.00

(IVP) for a spatially periodic unknown vector-valued function $u(t, x) \in \mathbf{R}^n$ of the time–space variable $(t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^N$:

$$\frac{\partial u}{\partial t} = D(t)\, \Delta u + F(t, u), \qquad t > 0,\ x \in \mathbf{R}^N; \tag{1}$$

$$u(t, x + \Omega_i e_i) = u(t, x), \qquad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^N,\ i = 1, ..., N; \tag{1_b}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbf{R}^N. \tag{1_i}$$

Here $\mathbf{R}^1_+ = [0, \infty)$, n and N are positive integers, $\Omega = (\Omega_1, ..., \Omega_N) \in \mathbf{R}^N$ is a spatial period with positive entries, and the vectors $e_i = (0, ..., 0, 1, 0, ..., 0)$, $i = 1, ..., N$, form the natural orthonormal basis in $\mathbf{R}^N$. The unknown function $u\colon \mathbf{R}^1_+ \times \mathbf{R}^N \to \mathbf{R}^n$ is a vector $u = (u_1, ..., u_n)$ satisfying the IVP above, $D(t) = \operatorname{diag}(d_1(t), ..., d_n(t))$ is a diagonal matrix with positive entries depending only on time, and $F(t, \cdot)\colon \mathbf{R}^n \to \mathbf{R}^n$ is an irreducible cooperative C^1-vector field, i.e., its Jacobian matrix $(\partial F_j/\partial u_k)^n_{j,k=1}$ is irreducible with $\partial F_j/\partial u_k \geqslant 0$ whenever $j \neq k$, for all $(t, u) \in \mathbf{R}^1_+ \times \mathbf{R}^n$. We assume that both $D(t)$ and $F(t, \cdot)$ are *τ-periodic* in time t with a period $\tau > 0$. Of course, Δ denotes the Laplacian.

Given any initial distribution $u_0 \in C(\mathbf{R}^N \to \mathbf{R}^n)$ which is spatially Ω-periodic, in order to guarantee global existence and uniqueness of a *classical* solution $u \in C(\mathbf{R}^1_+ \times \mathbf{R}^N \to \mathbf{R}^n) \cap C^{1,2}((0, \infty) \times \mathbf{R}^n)$ to the IVP above (cf. [2, 3] or [17]), we make the following two additional hypotheses:

(i) For some $\mu \in (0, 1)$, the function $D\colon \mathbf{R}^1_+ \to \mathbf{R}^{n \times n}$ is μ-Hölder continuous, and all mappings $F\colon \mathbf{R}^1_+ \times \mathbf{R}^n \to \mathbf{R}^n$ and $\partial F_j/\partial u_k$ are continuous and μ-Hölder continuous with respect to $t \in \mathbf{R}^1_+$ uniformly in compact subsets of $\mathbf{R}^1_+ \times \mathbf{R}^n$.

(ii) The corresponding system of ODEs

$$\frac{dv}{dt} = F(t, v), \qquad t > 0; \tag{2}$$

$$v(0) = v_0, \tag{2_i}$$

possesses a global solution $v \in C^1(\mathbf{R}^1_+ \to \mathbf{R}^n)$, for every $v_0 \in \mathbf{R}^n$.

Under these hypotheses the IVP generates a *dynamical process* in the underlying Banach space $V = C_{\Omega\text{-per}}(\mathbf{R}^N \to \mathbf{R}^n)$ of all continuous vector-valued functions $f\colon \mathbf{R}^n \to \mathbf{R}^n$ which are spatially Ω-periodic with the maximum norm $\|f\|_V = \max_{j,x} |f_j(x)|$. This process is *$\tau$-periodic* in time (cf. [8, Sect. 3.6]) and is defined globally in time since it is also *increasing*,

$$u(s) \leqslant \tilde{u}(s) \Rightarrow u(s+t) \leqslant \tilde{u}(s+t) \qquad \text{for all} \quad s, t \in \mathbf{R}^1_+,$$

whenever u, $\tilde{u}\colon \mathbf{R}^1_+ \to V$ are two classical solutions of the IVP with the initial distributions u_0, $\tilde{u}_0 \in V$, respectively. Here "$\leqslant$" denotes the *pointwise ordering* in V, i.e., $f \leqslant g$ in V means $f_j(x) \leqslant g_j(x)$ for all j and $x \in \mathbf{R}^N$. This monotonicity yields obvious lower and upper à priori bounds for a solution u of the IVP in terms of suitable solutions of the system of ODEs above (which are spatially constant). Moreover, using the strong maximum and boundary point principles (cf. [24, Chap. 3, Sect. 8]) it can be easily deduced from results in [14, Thm. 5.4] that our process is even *strongly increasing*,

$$u(s) < \tilde{u}(s) \Rightarrow u(x+t) \ll \tilde{u}(s+t) \qquad \text{for all} \quad s, t \in \mathbf{R}^1_+,$$

whenever u and $\tilde{u}$ are solutions of the IVP. Here $f < g$ in V means $f \leqslant g$ and $f \neq g$ (i.e., $f_j(x) \neq g_j(x)$ for some j and $x \in \mathbf{R}^N$), whereas $f \ll g$ means $g - f \in \operatorname{Int}(V_+)$, the interior of the positive cone $V_+ = \{f \in V\colon f \geqslant 0\}$ of V.

In particular, the *Poincaré* (or *period*) *map* $T\colon V \to V$ associated with our τ-periodic process is continuous and strongly increasing. Recall that if $u_0 \in V$ and $u\colon \mathbf{R}^1_+ \to V$ is the solution of the IVP, then $Tu_0 = u(\tau)$ by definition. Since $T^k u_0 = u(k\tau)$ for all $k \in \mathbf{Z}_+$, we will use the discrete-time semigroup $\{T^k\colon k \in \mathbf{Z}_+\}$ to investigate the asymptotic behavior of $u(t)$ as $t \to \infty$. Here $\mathbf{Z}_+ = \{0, 1, 2, ...\}$. Analogously, we denote by $V_\#$ the n-dimensional subspace of V consisting of all spatially constant functions $f \in V$, i.e., $V_\# \simeq \mathbf{R}^n$, and set $T_\# = T_{|V_\#}\colon V_\# \to V_\#$, the restriction of T to $V_\#$ which we identify with the Poincaré map associated with (2), the corresponding system of ODEs. Observe that $u_0 \in V_\# \Rightarrow u(t) \in V_\#$ for all $t \in \mathbf{R}^1_+$, where $u\colon \mathbf{R}^1_+ \to V$ is the solution of the IVP, by Hirsch [14, Proof of Thm. 5.4]. Thus, if $u_0 = v_0 \in V_\#$ we may identify u with the corresponding solution v of (2).

Given any $u_0 \in V$, we denote by $\mathcal{O}^+(u_0) = \{T^k u_0\colon k \in \mathbf{Z}_+\}$ its *positive semiorbit* (shortly, *orbit*), and by $\omega(u_0) = \{f \in V\colon T^{k_m} u_0 \to f (m \to \infty)$ for some sequence $k_m \to \infty$ in $\mathbf{Z}_+\}$ its *ω-limit set.* Our main result for the IVP is that the ω-limit set $\omega(u_0)$ of "almost every" initial distribution $u_0 \in V$ consists of spatially constant functions only, i.e., $\omega(u_0) \subset V_\#$, and consequently, the asymptotic behavior of the solution $u(t)$ to the IVP as $t \to \infty$ is determined by the dynamics on $\omega(u_0)$ induced by (2), the corresponding system of ODEs. It is usually a far simpler task to investigate the large-time asymptotic behavior of this system of ODEs, which is cooperative and irreducible, than that of the IVP, cf. [8]. A precise statement of our main result is as follows.

THEOREM 0.1. *Assume that the orbit $\mathcal{O}^+(v_0)$ of every $v_0 \in V_\#$ is bounded in $V_\#$. Then we have the following two statements:*

(a) *$\omega(u_0) \subset V_{\#}$ for all $u_0 \in V \setminus \mathcal{U}_2 \equiv \mathcal{S}_{1/2}$, where the set $\mathcal{U}_2 \subset V$ is characterized by $u_0 \in \mathcal{U}_2$ if and only if there exist u_0^1, $u_0^2 \in V$, $u_0^1 \ll u_0 \ll u_0^2$, such that both $\omega(u_0^1)$ and $\omega(u_0^2)$ are cycles in $V_{\#}$, say $\omega(u_0^i) = \mathcal{O}^+(p^i)$, $i = 1, 2$, for some $p^1 \ll p^2$ in $V_{\#}$, and $\omega(f^1) = \mathcal{O}^+(p^1)$ and $\omega(f^2) = \mathcal{O}^+(p^2)$ whenever $u_0^1 \leqslant f^1 < u_0 < f^2 \leqslant u_0^2$ in V.*

(b) *Moreover, $\mathcal{U}_2$ is contained in a union of at most countably many Lipschitz manifolds of codimension one in V. In particular, $\mu(\mathcal{U}_2) = 0$ for every Gaussian measure μ on V.*

The elements of $\mathcal{U}_2$ are called *ω-biunstable points* for T, whereas $\mathcal{S}_{1/2}$ is the set of all *ω-semistable points.* Our ω-stability notions coincide with those of Ljapunov stability in V as we will show in the next section. Combining Theorem 0.1 with a result of Hale and Somolinos [9, Thm. 4.2] for a cooperative system of two ODEs (whose proof is valid also under our hypotheses), and with recent results of Takáč [25, Prop. 1.1 and 1.2], we will be able to prove

COROLLARY 0.2. *Let all hypotheses from Theorem 0.1 be satisfied. Assume $u_0 \in \mathcal{S}_{1/2}$. If $n = 1$ then $\omega(u_0)$ is a single fixed point of T belonging to $V_{\#} \simeq \mathbf{R}^1$. If $n = 2$ then $\omega(u_0)$ is contained in a Lipschitz curve in $V_{\#} \simeq \mathbf{R}^2$ which is invariant under T, and $\omega(v_0)$ is a single fixed point of T for each $v_0 \in \omega(u_0)$.*

In other words, if $n = 1$ and $u_0 \in \mathcal{S}_{1/2}$, then the solution $u(t)$ of the IVP is asymptotic to the spatially constant τ-periodic solution $v(t)$ of the corresponding single ODE with $\omega(u_0) = \{v_0\}$, i.e.,

$$\|u(t + k\tau) - v(t)\|_V \to 0 \qquad \text{as} \quad k \to \infty \text{ in } \mathbf{Z}_+, \text{ uniformly for } t \in \mathbf{R}^1_+.$$

For the case $n = N = 1$ and *any* $u_0 \in V$, this result has been obtained by Chen and Matano [5]. If $n = 2$ and $u_0 \in \mathcal{S}_{1/2}$, then $u(t)$ is asymptotic to the compact set $\{v(t) : v_0 \in \omega(u_0)\}$ of spatially constant solutions of the corresponding system of two ODEs, where each $v(t)$ is asymptotic to a τ-periodic solution of that system.

Our proof of Theorem 0.1 is based on some ideas from Matano [18, Sect. 5] and Mierczyński and Poláčik [21] (who studied group actions on strongly monotone continuous-time semiflows generated by autonomous evolution equations with some spatial symmetry) combined with some techniques from Takáč [25] for strongly monotone discrete-time semigroups. More precisely, let $\mathcal{G}$ denote the group of all translations $g_a : \mathbf{R}^N \to \mathbf{R}^N$, $g_a(x) = x + a$, by a vector $a \in \mathbf{R}^N$. It follows from Hirsch [14, Proof of Thm. 5.4] that the action of $\mathcal{G}$ commutes with the process generated by the IVP, and in particular with T:

$$T(u_0 \circ g_a) = (Tu_0) \circ g_a \qquad \text{for all} \quad a \in \mathbf{R}^N \text{ and } u_0 \in V. \tag{3}$$

Here "$\circ$" denotes the composition of mappings. Now observe that V can be identified with the space of all continuous $\mathbf{R}^N$-valued functions defined on the N-dimensional torus $\mathbf{T}_\Omega^N = \mathbf{R}^N/(\Omega_1\mathbf{Z}\times\cdots\times\Omega_N\mathbf{Z})$, and therefore the action of $\mathscr{G}$ can also be identified with the action of the torus $G=\mathbf{T}_\Omega^N$, a compact connected metrizable group. For this reason we devote this article primarily to a proof of the following result; our terminology and notation will be clarified in the next section.

THEOREM 0.3. *Assume that V is a separable strongly ordered Banach space, X is a nonempty, open, and order-convex subset of V, and $T: X\to X$ is continuous, strongly increasing and ω-compact in every closed order interval $[a, b]$ in X. Let G be a compact connected metrizable group whose action on X, $x\mapsto g\cdot x$ for $g\in G$, is increasing and commutes with T. Then the following two statements are valid:*

(a) *For each $x\in\mathscr{S}_{1/2}\equiv X\backslash\mathscr{U}_2$, all points $w\in\omega(x)$ are symmetric: $g\cdot w=w$ for all $g\in G$.*

(b) *Furthermore, the set $\mathscr{U}_2$* is *"very small"; it is contained in a union of at most countably many Lipschitz manifolds of codimension one in $\hat{V}=(V, \|\cdot\|_e)$, the space V with the order topology, where each manifold is invariant under T and the action of G. In particular, $\mu(\mathscr{U}_2)=0$ for every Gaussian measure μ on V. The same statement is valid also for $\mathscr{U}_-$ and $\mathscr{U}_+$.*

The first investigation of the group action of a subgroup of $SO(N)$ on a strongly monotone continuous-time semiflow generated by an autonomous reaction–diffusion equation with some spatial symmetry was carried out by Matano [18, Sect. 5]. In fact, our Theorem 0.3 generalizes his [18, Thm. 5.2] and a similar result due to Mierczyński and Poláčik [21, Thm. 2.4]. In addition to this result for $\mathscr{S}_{1/2}$, Matano [19] also investigated $\mathscr{U}_2$ in his setting with autonomous reaction–diffusion equations.

This article is organized as follows: In Section 1 we introduce our basic concepts and present a few simple results; some of them from [13, 25]. In Section 2 we use these results to prove Theorem 2.1 which is a slightly more general version of Part (a) of our Theorems 0.3 and 0.1, and to prove Corollary 0.2 as well. In Section 3 we prove Part (b) of Theorems 0.3 and 0.1. In Section 4 we present some more examples to which Theorem 0.3 can be applied. Finally, Section 5 contains a few concluding remarks.

1. Basic Concepts and Preliminary Results

We start with some notation and a few definitions. Throughout the entire paper we assume the following three hypotheses (X), (V), and (T):

(X): X is an ordered, metrizable topological space, i.e., X is a metrizable topological space with a closed (partial) order relation "$\leqslant$" in $X \times X$ (shortly, X is an *ordered space*). We write $x \ll y$ if (x, y) belongs to the interior of the order relation in $X \times X$, while $x < y$ means $x \leqslant y$, $x \neq y$.

(V): V is a strongly ordered, metrizable topological vector space (shortly, *strongly ordered vector space*), which is equivalent to saying that the positive cone $V_+ = \{x \in V: x \geqslant 0\}$ of V has nonempty interior denoted by $\mathrm{Int}(V_+)$. (In some of our results we will assume that X is a nonempty subset of V with closure $\mathrm{Cl}(X)$.)

(T): T is a continuous, strongly increasing mapping of X into itself, i.e., $x, y \in X$ and $x < y$ implies $Tx \ll Ty$.

An ordered space X is called *strongly ordered* if every open subset U of X satisfies:

$(SO1)$ If $x \in U$ then $a \ll x \ll b$ for some $a, b \in U$.

It is easy to see that, for every open subset U of X, $(SO1)$ implies:

$(SO2)$ If $a, b \in U$ and $a \ll b$ then $a \ll x \ll b$ for some $x \in U$.

For example, every nonempty open subset of V is a strongly ordered space.

The *positive semiorbit* (shortly, *orbit*) of any $x \in X$ is defined by $\mathcal{O}^+(x) = \{T^n x: n \in \mathbf{Z}_+\}$, and the *$\omega$-limit set* of x is defined by $\omega(x) = \{y \in X: T^{n_k} x \to y (k \to \infty)$ for some sequence $n_k \to \infty$ in $\mathbf{Z}_+\}$. Note that if $\mathcal{O}^+(x)$ is relatively compact in X, then $\omega(x) \neq \varnothing$. A subset Y of X is called *positively invariant* (shortly, *invariant*) if $T(Y) \subset Y$, and *totally invariant* if $T(Y) = Y$. For instance, every $\mathcal{O}^+(x)$ is invariant, and every $\omega(x)$ with $\mathcal{O}^+(x)$ relatively compact is totally invariant.

Given $a, b \in X$, the set $[a, b] = \{x \in X: a \leqslant x \leqslant b\}$ is called a *closed order interval*, and $[\![a, b]\!] = \{x \in X: a \ll b\}$ is called an *open order interval* in X. We write $[a, \infty]\!] = \{x \in X: x \geqslant a\}$, and similarly for $[\![-\infty, b]$, etc. A subset Y of X is called *order-convex* in X if $[a, b] \subset Y$ whenever $a, b \in Y$ and $a < b$; *lower closed* if $[\![-\infty, b] \subset Y$ whenever $b \in Y$; *upper closed* if $[a, \infty]\!] \subset Y$ whenever $a \in Y$; and *unordered* if no pair of points $x, y \in Y$ satisfies $x < y$.

We denote closed order intervals in V by $[a, b]_V = \{x \in V: a \leqslant x \leqslant b\}$, and similarly, all other concepts in V will be marked by the subscript V in case confusion might arise.

If X is a strongly ordered space we define the *order topology* on X whose neighborhood base is generated by all open order intervals $[\![a, b]\!]$ with $a \ll b$. If $Y \subset X$, we denote by $\hat{Y}$ the set Y endowed with the induced order topology. A subset Y of X is called *order-open* (*order-closed*, resp.) if it is open (closed, resp.) in $\hat{X}$. Note that the identity mapping $\hat{\imath}: X \to \hat{X}$ is continuous, but in general not homeomorphic. It is easy to see that the order topology on V is induced by any *ordered norm* $|\cdot|_e$ on V defined by

$$|x|_e = \inf\{\lambda \in \mathbf{R}^1_+ : -\lambda e \leqslant x \leqslant \lambda e\} \qquad \text{for some} \quad e \in \operatorname{Int}(V_+). \tag{4}$$

It is proved in [13, 14] that if $T: X \to X$ is continuous and increasing $(x \leqslant y \Rightarrow Tx \leqslant Ty)$, then also $T: \hat{X} \to \hat{X}$ is continuous in which case we say that T is *order continuous.*

We will need the following two elementary convergence results proved in [13] (cf. also [25]). We denote by $\mathscr{E}(T) = \{x \in X: Tx = x\}$ the set of all *equilibria* (i.e., *fixed points*) of T. Given $k \in \mathbf{N}$, the elements of $\mathscr{E}(T^k)$ are called *k-periodic points* of T. The orbit $\mathcal{O}^+(x)$ of a k-periodic point $x \in \mathscr{E}(T^k)$ is called a *k-cycle.*

Proposition 1.1 (Convergence Criterion for Strongly Monotone Semigroups). *Let X and T satisfy (X) and (T). Assume that $x \in X$, $\mathcal{O}^+(x)$ is relatively compact, and either $T^k x > x$ or $T^k x < x$ for some $k \in \mathbf{N}$. Then $T^{nk+l}x \to T^l p$ as $n \to \infty$, $l = 0, 1, ..., k-1$, for some $p \in \mathscr{E}(T^k)$, and either $p \gg x$ or $p \ll x$, respectively. Moreover, $\omega(x)$ is a k-*cycle.

Proposition 1.2 (Nonordering of Limit Sets). *Let X and T satisfy (X) and (T). Assume that $x \in X$ and $\mathcal{O}^+(x)$ is relatively compact. Then $\omega(x)$ is nonempty and unordered. If* $\operatorname{Cl}(\mathcal{O}^+(x))$ *is not unordered, then $\omega(x)$ is a cycle.*

It is convenient to introduce the following ordering "$\preccurlyeq$" of subsets of X: If A, $B \subset X$, we write $A \preccurlyeq B$ if and only if

$$A \subset B_- \equiv \bigcup \{[\![-\infty, x]\!]: x \in B\} \qquad \text{and} \qquad B \subset A_+ \equiv \bigcup \{[\![x, \infty]\!]: x \in A\}.$$

We write $A \prec B$ for $A \preccurlyeq B$ and $A \neq B$, while $A \lll B$ means $A \subset \operatorname{Int}(B_-)$ and $B \subset \operatorname{Int}(A_+)$. Observe that $x \leqslant y$ in X with $\mathcal{O}^+(x)$ and $\mathcal{O}^+(y)$ relatively compact implies $\omega(x) \preccurlyeq \omega(y)$, by the monotonicity of T. If also $\omega(x) \cap \omega(y) = \varnothing$ then $\omega(x) \lll \omega(y)$, since T is strongly increasing.

Throughout the remaining part of this paper we assume that X and T satisfy (X) and (T), and X is strongly ordered. We say that the mapping T is *ω-compact* in a subset Y of X if $\mathcal{O}^+(x)$ is relatively compact for each $x \in Y$, and also $\bigcup_{x \in Y} \omega(x)$ is relatively compact in X. From now on we will always assume that T is ω-compact in $[a, b]$ for all a, $b \in X$ with $a \leqslant b$, in which case we define the *lower* and *upper ω-limit sets* of $x \in X$ by

$$\omega_-(x) = \bigcap_{\substack{y \in X \\ y \ll x}} \operatorname{Cl} \bigcup_{\substack{z \in X \\ y \leqslant z < x}} \omega(z) \qquad \text{and} \qquad \omega_+(x) = \bigcap_{\substack{y \in X \\ y \gg x}} \operatorname{Cl} \bigcup_{\substack{z \in X \\ y \geqslant z > x}} \omega(z),$$

respectively, cf. [26]. Observe that both $\omega_-(x)$ and $\omega_+(x)$ are compact, invariant, and nonempty, since T is ω-compact in every $[a, b] \in X$ and there exist sequences a_n, $b_n \in X$, $a_n < x < b_n$, such that $a_n \to x$ and $b_n \to x$ as $n \to \infty$. It is also easy to obtain $\omega_-(x) \preccurlyeq \omega(x) \preccurlyeq \omega_+(x)$ from $\omega(z_*) \preccurlyeq \omega(x) \preccurlyeq \omega(z^*)$ for $z_* < x < z^*$ in X.

Under this ω-compactness hypothesis for T we are able to describe some useful continuity properties of the set-valued mapping $\omega: X \to X$ and to introduce the following stability classification of an arbitrary point $x \in X$ as well:

We say that x is *lower (upper) ω-stable* if $\omega_-(x) = \omega(x) (\omega_+(x) = \omega(x)$, resp.); otherwise x is *lower (upper) ω-unstable.* We say that x is *lower (upper) asymptotically ω-stable* if there exists $y \in X$, $y < x$ ($y > x$, resp.) such that $\omega(y) = \omega(x)$. The set of all lower (upper) ω-stable points $x \in X$ is denoted by $\mathscr{S}_-$ ($\mathscr{S}_+$), the set of all lower (upper) ω-unstable points by $\mathscr{U}_-$ ($\mathscr{U}_+$), and the set of all lower (upper) asymptotically ω-stable points by $\mathscr{A}_-$ ($\mathscr{A}_+$). Finally, we denote by $\mathscr{S}_{1/2} = \mathscr{S}_- \cup \mathscr{S}_+$ the set of all *ω-semistable* points, by $\mathscr{U}_2 = \mathscr{U}_- \cap \mathscr{U}_+$ the set of all *ω-biunstable* points, and by $\mathscr{A}_{1/2} = \mathscr{A}_- \cup \mathscr{A}_+$ the set of all *asymptotically ω-semistable* points.

Observe that our stability notions are equivalent to the continuity properties of the set-valued mapping $\omega: X \to X$. Furthermore, if X is an open subset of V where V satisfies (V), then our ω-stability coincides with the Ljapunov stability in $\hat{V}$ endowed with any ordered norm $|\cdot|_e$ on V defined by (4):

PROPOSITION 1.3 (Ljapunov Stability). *Let X, V, and T satisfy* (X), (V), *abd* (T), *and let X be a nonempty open subset of V. Assume that T is ω-compact in every closed order interval $[a, b]$ in X. Let $x \in X$. Then $x \in \mathscr{S}_-$ if and only if x is* lower order Ljapunov stable, *i.e., for any fixed $e \in \operatorname{Int}(V_+)$ the following statement holds*:

(LOS) given any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in X$ with $y \leqslant x$ we have

$$|x - y|_e \leqslant \delta \Rightarrow |T^k x - T^k y|_e \leqslant \varepsilon \qquad \textit{for all} \quad k \in \mathbf{Z}_+.$$

A corresponding result with reversed ordering holds for $\mathscr{S}_+$, i.e., $x \in \mathscr{S}_+$ if and only if $x \in X$ is upper order Ljapunov stable.

Proof. $x \in \mathscr{S}_- \Rightarrow$ (LOS). On the contrary, suppose $x \in \mathscr{S}_-$, but (LOS) is false. Then there exist $\varepsilon > 0$ and two sequences $\{x_n\} \subset X$ and $\{k_n\} \subset \mathbf{Z}_+$ such that

$$x_n < x,\ |x - x_n|_e \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad |T^{k_n} x - T^{k_n} x_n|_e \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

The order continuity of T forces $k_n \to \infty$, and so we may assume $1 \leqslant k_1 < k_2 < \cdots$ by passing to a subsequence. Since $Tx_n \ll Tx$ and $|Tx - Tx_n|_e \to 0$ as $n \to \infty$, we may assume also $Tx_1 \ll Tx_2 \ll \cdots \ll Tx$ by passing to another subsequence. Now fix any $m \in \mathbf{N}$. Then $T^{k_n}x_m \leqslant T^{k_n}x_n \ll T^{k_n}x$ for all $n \geqslant m$, which shows that $v_m \leqslant w$ for some $v_m \in \omega(x_m)$ and $w \in \omega(x)$. Moreover, $|w - v_m|_e \geqslant \varepsilon$ follows from $|T^{k_n}x - T^{k_n}x_m|_e \geqslant |T^{k_n}x - T^{k_n}x_m|_e \geqslant \varepsilon$ for all $n \geqslant m$. Since T is ω-compact in $Y = \{x, x_1, x_2, ...\}$, the sequence $\{v_m\}_{m=1}^{\infty}$ contains a subsequence which converges in X, and hence, also in $\hat{X}$ to some $v \in \omega_-(x) = \omega(x)$. Moreover, $|w - v|_e \geqslant \varepsilon$ follows from $|w - v_m|_e \geqslant \varepsilon$ for all $m \geqslant 1$. Thus $v < w$ in $\omega(x)$ which is impossible since $\omega(x)$ is unordered by Proposition 1.2. We have proved $x \in \mathscr{S}_- \Rightarrow$ (LOS).

(LOS) $\Rightarrow x \in \mathscr{S}_-$. Fix any $v \in \omega_-(x)$. Then there exist two sequences $\{x_n\} \subset X$ and $v_n \in \omega(x_n)$ such that $x_n < x$, $x_n \to x$ and $v_n \to v$ in X as $n \to \infty$. Applying (LOS) we arrive at $|T^k x - T^k x_n|_e \to 0$ as $n \to \infty$ uniformly in $k \in \mathbf{Z}_+$. Thus, there exists a sequence $w_n \in \omega(x)$ such that $v_n \leqslant w_n$ and $|w_n - v_n|_e \to 0$ as $n \to \infty$. Since $\omega(x)$ is compact in X, we can achieve also $w_n \to w \in \omega(x)$ as $n \to \infty$ by passing to a subsequence. Consequently, $|w_n - v_n|_e \to 0$ as $n \to \infty$ shows that $v = w \in \omega(x)$. We have proved $\omega_-(x) \subset \omega(x)$. But then $\omega_-(x) \preccurlyeq \omega(x)$ and $\omega(x)$ unordered force $\omega_-(x) = \omega(x)$, i.e., $x \in \mathscr{S}_-$ as desired. ▮

The structure of the ω-limit sets near an ω-unstable point $x \in X$ is very simple:

PROPOSITION 1.4 (Discontinuity Principle). *Let X and T satisfy (X) and (T), and let X be strongly ordered. Assume that T is ω-compact in every closed order interval $[a, b]$ in X. Let $x \in X$ be arbitrary. Then $x \in \mathscr{U}_-$ if and only if $\omega_-(x)$ is a cycle satisfying $\omega_-(x) \lll \omega(x)$. If this is the case then there exists $a \in X$, $a \ll x$, such that $\omega(y) = \omega_-(x)$ for every $y \in X$ with $a \leqslant y < x$, and also $\omega(y) = \omega_-(x)$ for every $y \in Y_0$, where*

$$Y_0 = \{y \in X\colon p < y < q \text{ for some } p \in \omega_-(x) \text{ and } q \in \omega(x)\}.$$

A corresponding result holds for the set $\mathscr{U}_+$.

Proof. First assume $x \in \mathscr{U}_-$. Hence $\omega_-(x) \prec \omega(x)$, and so we have $v < w$ for some $v \in \omega_-(x)$ and $w \in \omega(x)$. Making use of $Tv \ll Tw$ with $Tv \in \omega_-(x)$ and $Tw \in \omega(x)$ we may assume $v \ll w$. Since $w \in \omega(x)$, choose first $k \in \mathbf{Z}_+$ such that $v \ll T^k x$, and then $a \in X$, $a \ll x$, such that $v \ll T^k a$. Now fix any $y \in X$, $a \leqslant y < x$. Since $Ty \ll Tx$ and $v \in \omega_-(x)$, choose $z \in X$, $a \leqslant z < x$, and $m \in \mathbf{Z}_+, m > k$, such that $Ty \leqslant Tz$ and $T^m z \ll T^k a$. Then $T^m z \ll (T^k a \leqslant)\, T^k z$, and we may apply Proposition 1.2 to conclude that $\omega(z)$ is a cycle. Furthermore, $a \leqslant z$ and $T^m z \ll T^k a$ imply $\omega(a) \preccurlyeq \omega(z) \preccurlyeq$

$\omega(a)$ which, in turn, forces $\omega(z) = \omega(a)$ since $\omega(a)$ is unordered. From $Ta \leqslant Ty \leqslant Tz$ we obtain also $\omega(y) = \omega(a)$. Consequently, $\omega(y) = \omega_-(x) = \omega(a)$ is a cycle for every $y \in X$ with $a \leqslant y < x$.

Conversely, if $\omega_-(x)$ is a cycle satisfying $\omega_-(x) \lll \omega(x)$, then obviously $x \in \mathcal{U}_-$.

Finally, let $x \in \mathcal{U}_-$ and fix any $y \in Y_0$, i.e., $p < y < q$ for some $p \in \omega_-(x)$ and $q \in \omega(x)$. By $Ty \ll Tq \in \omega(x)$ we can choose first $k \in \mathbf{Z}_+$ such that $Ty \ll T^k x$, and then $z \in X, a \leqslant z < x$, such that $Ty \leqslant T^k z$. It follows that $\omega(p) \preccurlyeq \omega(z) = \omega_-(x)$. Since $p \in \omega_-(x)$, where $\omega_-(x)$ is a cycle, we conclude that $\omega(y) = \omega_-(x)$ for every $y \in Y_0$. In particular, $\omega_-(x) \cap \omega(x) = \varnothing$ entails $\omega_-(x) \lll \omega(x)$ since T is strongly increasing. ∎

2. Proof of Theorem 0.3(a)

In addition to our hypotheses (X), (V), and (T) in Section 1, from now on we will assume also

(G): G is a compact connected metrizable topological group (shortly, *a compact connected group*) acting on X in such a way that its action is increasing and commutes with T.

A mapping $\gamma: G \times X \to X$ is called a *group action* of G on X if it is jointly continuous and $g \mapsto \gamma(g, \cdot) \equiv \Gamma(g)$ is a group homomorphism of G into $\mathrm{Hom}(X)$, the group of all homeomorphisms of X onto itself. We say that γ is *increasing* if, for each $g \in G$, the mapping $\Gamma(g): X \to X$ is increasing, i.e., $x \leqslant y$ in X implies $\Gamma(g)\, x \leqslant \Gamma(g)\, y$. Finally, we say that γ *commutes* with T if, for each $g \in G$, the mapping $\Gamma(g)$ commutes with T. We write $\gamma(g, x) \equiv g \cdot x$ and identify $\Gamma(g) \equiv g$, for brevity, as we may since we consider only one action at the time. The unit element of G is denoted by e. If $S \subset G$ and $Y \subset X$, we write $S \cdot Y = \gamma(S \times Y) = \{g \cdot x: (g, x) \in S \times Y\}$. We say that a subset Y of X is *invariant under* γ if $G \cdot Y \subset Y$, while $x \in X$ is called *symmetric* if $g \cdot x = x$ for all $g \in G$.

The following theorem is a more general version of Part (a) of our main result, Theorem 0.3, and of Theorem 0.1 as well:

THEOREM 2.1. *Let X, T, and G satisfy (X), (T), and (G), and let X be strongly ordered. Assume that T is ω-compact in every closed order interval $[a, b]$ in X. If $x \in \mathcal{S}_{1/2}$ then every $w \in \omega(x)$ is symmetric.*

For strongly increasing continuous-time semiflows, the symmetry statement for $x \in \mathcal{S}_{1/2}$ is due to Mierczyński and Poláčik [21, Thm. 2.4], and for the group action of a subgroup of $SO(N)$ on an autonomous reaction–diffusion equation with some spatial symmetry it is due to Matano [18, Thm. 5.2]. Similarly to [21], the proof of our theorem uses the fact that the topology on G can be induced by a left-invariant metric ρ, i.e., $\rho(ag, ah) = \rho(g, h)$ for all $a, g, h \in G$, cf. [22, Sect. 1.22]. Namely, if σ is a metric for G, define $\rho(g, h) = \max_{a \in G} \sigma(ag, ah)$ to obtain a left-invariant metric ρ for G. From now on, ρ denotes such a metric for G. With the help of ρ, Mierczyński and Poláčik [21, Lemma 1.2] showed

LEMMA 2.2. *Let G be a compact metrizable topological group with unit element e. Then for every $g \in G$ there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subset N$ such that $g^{n_k} \to e$ as $k \to \infty$.*

Proof of Theorem 2.1. Since $x \in \mathscr{S}_{1/2}$, we may assume $x \in \mathscr{S}_-$, the case $x \in \mathscr{S}_+$ being analogous. We distinguish between the following two cases:

Case (i). $x \in \mathscr{A}_-$.

Thus $\omega(x) = \omega(y)$ for some $y \ll x$ in X. Choose $\delta > 0$ such that $y \ll g \cdot x$ for all $g \in U_e \equiv \{g \in G: \rho(g, e) < \delta\}$. More generally, for any g, $h \in G$ we have

$$h \cdot y \ll g \cdot x \quad \Leftrightarrow \quad y \ll h^{-1} g \cdot x,$$

since $\Gamma(h)$ being increasing and homeomorphic preserves the strong ordering in X, for each $h \in G$, and consequently $h \cdot y \ll g \cdot x$ whenever $h^{-1} g \in U_e$, i.e., $g \in U_h \equiv hU_e$. Clearly, $\{U_h: h \in G\}$ forms an open cover of G, because each $U_h = \{g \in G: \rho(g, h) < \delta\}$ is open. The compactness of G guarantees the existence of a finite subcover $\{U'_i \equiv U_{h_i}: i = 1, ..., m\}$ of G. Now fix any $\tilde{g} \in G$. We can always permute the sets U'_i, $1 \leqslant i \leqslant m$, so that $e \in U'_1$, $\tilde{g} \in U'_j$ for some $j \in \{1, ..., m\}$, and there exist $g_i \in U'_i \cap U'_{i+1} \neq \emptyset$ for all $i \in \{1, ..., j-1\}$, since G is connected. Set $g_0 = e$ and $g_j = \tilde{g}$. We obtain

$$h_i \cdot y \ll g_{i-1} \cdot x \quad \text{and} \quad h_i \cdot y \ll g_i \cdot x \quad \text{for} \quad 1 \leqslant i \leqslant j. \tag{5}$$

Now consider any $w \in \omega(x) = \omega(y)$. Since $y \ll x$ implies $T^n y \ll T^n x$ for all $n \in \mathbf{Z}_+$, there exists a sequence $n_k \in \mathbf{Z}_+$, $n_k \to \infty$, such that $T^{n_k} x \to w$ and $T^{n_k} y \to w$. Since T commutes with γ, (5) entails

$$h_i \cdot w \leqslant g_{i-1} \cdot w \quad \text{and} \quad h_i \cdot w \leqslant g_i \cdot w \quad \text{for} \quad 1 \leqslant i \leqslant j. \tag{6}$$

In general, suppose $w \leqslant g \cdot w$ for some $(g, w) \in G \times X$. Then $w \leqslant g \cdot w \leqslant g^2 \cdot w \leqslant \cdots$ is a sequence in $G \cdot w$, a compact set in X, and so we may apply Lemma 2.2 (as did Mierczyński and Poláčik [21, Prop. 1.3]) to conclude that $w \leqslant g \cdot w \leqslant w$, i.e., $g \cdot w = w$. It follows from (6) that $g_{i-1} \cdot w = h_i \cdot w = g_i \cdot w$ for $1 \leqslant i \leqslant j$, and hence $\tilde{g} \cdot w = w$ as desired.

Case (ii). $x \in \mathscr{S}_- \setminus \mathscr{A}_-$.

Pick any $y < x$ in X; so $\omega(y) \prec \omega_-(x) = \omega(x)$. Define the set

$$M = \{ g \in G \colon T^n y \ll g \cdot T^n x \text{ for some } n \in \mathbf{Z}_+ \}.$$

Clearly $e \in M$ by $Ty \ll Tx$, and M is open in G. To show that M is also closed in G, we fix any $g \in \mathrm{Cl}(M)$. Then we can find a sequence $g_k \in M$ with $g_k \to g$, and another sequence $n_k \in \mathbf{Z}_+$ with $T^{n_k} y \ll g_k \cdot T^{n_k} x$. Since T is strongly increasing, we may assume $n_1 < n_2 < \cdots$. Passing to a subsequence if necessary we obtain $T^{n_k} y \to v$ and $T^{n_k} x \to w$ as $k \to \infty$, for some $v \in \omega(y) \backslash \omega(x) \neq \varnothing$ and $w \in \omega(x)$, by $\omega(y) \prec \omega(x)$. Hence $v \leqslant g \cdot w$. Obviously $v \leqslant w$ since $y < x$, and so $v < w$. If $v = g \cdot w$ then $g \cdot w \leqslant w$, and so $\cdots \leqslant g^2 \cdot w \leqslant g \cdot w \leqslant w$ is a sequence in $G \cdot w$. Applying Lemma 2.2 we obtain $w \leqslant g \cdot w \leqslant w$ whence $v = g \cdot w = w$, a contradiction to $v < w$. Thus, we must have $v < g \cdot w$ which entails $Tv \ll g \cdot Tw$. Taking $k \in \mathbf{N}$ sufficiently large we arrive at $T^{n_k+1} y \ll g \cdot T^{n_k+1} x$ which proves $g \in M$. So M is closed. We conclude that $M = G$ since G is connected.

Now let $w \in \omega(x)$ and $g \in G$ be arbitrary. Hence, given any $y \in X$ with $y < x$, there exists a sequence $n_1 < n_2 < \cdots$ satisfying $T^{n_k} x \to w$, $T^{n_k} y \to v \in \omega(y)$ and $T^{n_k} y \ll g \cdot T^{n_k} x$ for all $k \in \mathbf{N}$. Similarly as above $v \leqslant w$ and $v \leqslant g \cdot w$. If $v = w$ or $v = g \cdot w$, then $w \leqslant g \cdot w$ or $g \cdot w \leqslant w$, respectively, and so $w \leqslant g \cdot w \leqslant g^2 \cdot w \leqslant \cdots$ or $\cdots \leqslant g^2 \cdot w \leqslant g \cdot w \leqslant w$ is a sequence in $G \cdot w$. Applying Lemma 2.2 again we obtain $w \leqslant g \cdot w \leqslant w$ whence $v = g \cdot w = w$ as desired. Thus, from now on we may assume $v < w$ and $v < g \cdot w$ for all $y \in X$ with $y < x$. Consequently, taking $y = z_n \nearrow x$ with $z_n < x$ for $n \in \mathbf{N}$ we can construct a sequence $v_n \in \omega(z_n)$ such that $v_n < w$ and $v_n < g \cdot w$. This sequence is relatively compact in X since T is ω-compact in $Y = \{x, z_1, z_2, \ldots\}$; let $u \in X$ be any limit point of $\{v_n\}_{n=1}^{\infty}$. Then $u \in \omega_-(x) = \omega(x)$ by $x \in \mathscr{S}_-$, and $u \leqslant w$ and $u \leqslant g \cdot w$. Thus, $\omega(x)$ unordered forces $u = w$, and so $w \leqslant g \cdot w$ which entails $g \cdot w = w$ as above. We have proved Theorem 2.1. ∎

Note that in the statements of Theorems 0.3(a) and 0.1(a) we have used an equivalent characterization of the set $\mathscr{U}_2$ stated in Proposition 1.4. In particular, if $x \in \mathscr{U}_-$ then $\omega_-(x)$ is a cycle in $\mathscr{A}_+ \subset \mathscr{S}_+$, and $\omega(y) = \omega_-(x)$ for every $y \in Y_0$, where

$$Y_0 = \{ y \in X \colon p < y < q \text{ for some } p \in \omega_-(x) \text{ and } q \in \omega(x) \}.$$

A corresponding result holds for $x \in \mathscr{U}_+$.

Proof of Theorem 0.3(a). This result follows directly from Theorem 2.1. ∎

Proof of Theorem 0.1(a). We consider the IVP (1) from the Introduction and assume $u_0 \in \mathscr{S}_{1/2}$. We recall our notation: $\mathscr{G}$ denotes the group of all translations $g_a: \mathbf{R}^N \to \mathbf{R}^N$, $g_a(x) = x + a$, by a vector $a \in \mathbf{R}^N$. The action of $\mathscr{G}$ on the discrete-time semigroup $\{T^k: k \in \mathbf{Z}_+\}$ is defined through the action of $\mathscr{G}$ on the space V by translation,

$$(g_a \cdot f)(x) \stackrel{\text{def}}{=} (f \circ g_a)(x) = f(x+a), \qquad x \in \mathbf{R}^N,$$

for all $a \in \mathbf{R}^N$ and $f \in V = C_{\Omega\text{-per}}(\mathbf{R}^N \to \mathbf{R}^n)$. By Hirsch [14, Proof of Thm. 5.4], the action of $\mathscr{G}$ commutes with the process generated by the IVP, and in particular with T. The action of $\mathscr{G}$ on V can be identified with the action of the torus $G = \mathbf{T}^N_\Omega = \mathbf{R}^N/(\Omega_1 \mathbf{Z} \times \cdots \times \Omega_N \mathbf{Z})$, a compact connected metrizable group, on $C_{\Omega\text{-per}}(\mathbf{R}^N \to \mathbf{R}^n)$.

By Theorem 0.3(a), every function $f \in \omega(u_0)$ is symmetric, i.e., $f(x+a) = f(x)$ for all x, $a \in \mathbf{R}^N$, and so $f \in V_\# \simeq \mathbf{R}^n$ as desired. ∎

Proof of Corollary 0.2. Again, consider the IVP and assume $u_0 \in \mathscr{S}_{1/2}$. We have $\omega(u_0) \subset V_\#$ by Theorem 0.1(a), and so the dynamics of T on $\omega(u_0)$ is given by $T_\# = T_{|V_\#}: V_\# \to V_\#$, the Poincaré map associated with the corresponding system of ODEs (2). Namely, it follows from [14, Proof of Thm. 5.4] that the solution $u: \mathbf{R}^1_+ \to V$ of the IVP satisfies $u_0 \in V_\# \Rightarrow u(t) \in V_\#$ for all $t \in \mathbf{R}^1_+$. Consequently, if $u_0 = v_0 \in V_\#$ we may identify u with the corresponding solution v of (2).

If $n = 1$ then $\omega(u_0)$ is an unordered invariant subset of $V_\# \simeq \mathbf{R}^1$, and hence, a single fixed point of T. If $n = 2$ then we employ a result due to Hale and Somolinos [9, Thm. 4.2] to conclude that $\omega(v_0)$ is a single fixed point of T for each $v_0 \in \omega(u_0)$. Furthermore, it follows from [25, Prop. 1.1 and 1.2] that $\omega(u_0)$ is contained in an invariant Lipschitz curve in $V_\# \simeq \mathbf{R}^2$. ∎

3. Proof of Theorem 0.3(b)

We start with the following two lemmas studying a generic point $x \in \mathscr{U}_-$:

Lemma 3.1. *Let X, V, T, and G satisfy* (*X*), (*V*), (*T*), *and* (*G*), *and let X be a nonempty, open, and order-convex subset of V. Assume that T is ω-compact in every closed order interval $[a, b]$ in X. Let $x \in \mathscr{U}_-$, and define the set*

$$Y = \{ y \in X: p \ll T^m u \ll q \text{ for some } p \in \omega_-(x), q \in \omega(x) \text{ and } m \in \mathbf{Z}_+ \}.$$

Then we have the following two statements:

(i) *Y is a nonempty, open, and order-convex subset of V; and*

(ii) *$T(Y) \subset Y$, $T^{-1}(Y) \subset Y$, and $g \cdot Y = Y$ for all $g \in G$.*

A corresponding result with reversed ordering holds for $x \in \mathscr{U}_+$.

Proof. (i) By Proposition 1.4 we have $\emptyset \neq Y_0 \subset Y$. The continuity of T implies that Y is open in V. To show that Y is order-convex in V we fix any $y, y' \in Y$ with $y \leqslant y'$. Thus, we have $p \ll T^m y \ll q$ and $p' \ll T^{m'} y' \ll q'$ for some $p, p' \in \omega_-(x)$, $q, q' \in \omega(x)$ and $m, m' \in \mathbf{Z}_+$. Since both $\omega_-(x)$ and $\omega(x)$ are invariant under T, we may replace m and m' by $\max\{m, m'\}$, and so we may assume $m = m'$. We arrive at $p \ll T^m y \leqslant T^m y' \ll q'$ which shows that $[y, y'] = [y, y']_V \subset Y$.

(ii) We have $T(Y) \subset Y$ by the invariance of $\omega_-(x)$ and $\omega(x)$, while $T^{-1}(Y) \subset Y$ follows from our definition of Y.

Now fix an arbitrary $g \in G$; we want to show $g \cdot Y = Y$. Pick any $y \in Y$. Then $p \ll T^m y \ll q$ for some $p \in \omega_-(x)$, $q \in \omega(x)$ and $m \in \mathbf{Z}_+$. From Proposition 1.4 we obtain $\omega_-(x) \subset \mathscr{A}_+ \subset \mathscr{S}_+$ which implies $g \cdot p = p$ by Theorem 2.1. Then $p \ll g^{-1} \cdot q$. By Proposition 1.4 we have also $\omega(y) = \omega_-(x)$, and hence, we may choose $m \in \mathbf{Z}_+$ such that $p \ll T^m y \ll g^{-1} \cdot q$. Consequently $p = g \cdot p \ll T^m(g \cdot y) \ll q$ since T and γ commute, and therefore $g \cdot y \in Y$ as desired. We have proved $G \cdot Y \subset Y$; so $g \cdot Y = Y$ for all $g \in G$, by $g, g^{-1}, e \in G$. ∎

Given an open and order-convex subset Y of X with the boundary ∂Y in X, we denote by

$$\partial_- Y = \{x \in \partial Y : x \ll y \text{ for some } y \in Y\}$$

the *lower boundary* of Y, and by

$$\partial_+ Y = \{x \in \partial Y : x \gg y \text{ for some } y \in Y\}$$

the *upper boundary* of Y.

LEMMA 3.2. *Let all hypotheses of Lemma* 3.1 *be satisfied. Let* $x \in \mathscr{U}_-$, *and define the set* Y *as in Lemma* 3.1. *Then we have the following three statements:*

(i) $Y \subset \mathscr{A} \subset \mathscr{S}$ *and* $x \in \partial_+ Y \subset \mathscr{U}_-$;

(ii) $T(\partial_+ Y) \subset \partial_+ Y$, *and* $g \cdot \partial_+ Y = \partial_+ Y$ *for all* $g \in G$; *and*

(iii) $\partial_+ Y$ *is an unordered set.*

A corresponding result holds for $x \in \mathscr{U}_+$.

Proof. (i) By Proposition 1.4 we have $\omega(y) = \omega_-(x) \lll \omega(x)$ for every $y \in Y$ which proves $Y \subset \mathscr{A} \subset \mathscr{S}$ and also $\omega_-(x) \subset \partial_- Y$ and $\omega(x) \subset \partial_+ Y$.

Now choose any $p \in \omega_-(x)$ and $q \in \omega(x)$ with $p \ll q$. Then $p \ll T^n x$ for some $n \in \mathbf{Z}_+$, and so there exists $a \in X$, $a \ll x$, such that $p \ll T^n a$. Combining this fact with Proposition 1.4 we arrive at $p \ll T^n y < T^n x$ and $\omega(y) = \omega_-(x)$ for avery $y \in X$ with $a \leqslant y < x$. Since $p \in \omega(y)$, we can find $m \in \mathbf{Z}_+$, $m \geqslant n$, such that $p \ll T^m y \ll q$, whence $y \in Y$. It follows that $x \in \partial_+ Y$.

To verify $\partial_+ Y \subset \mathscr{U}_-$ we fix an arbitrary $z \in \partial_+ Y$. Hence $y \ll z$ for some $y \in Y$, and there exists a sequence $\{z_n\}$ in Y such that $y \ll z_n \to z$ as $n \to \infty$. We have $[y, z_n] \subset Y$ since Y is order-convex in V, and so we can construct a sequence $y_n \in [y, z_n]$ such that $y \ll y_1 \ll y_2 \ll \cdots \ll z$ and $y_n \to z$ as $n \to \infty$. We already know that $\omega(y_n) = \omega_-(x) \lll \omega(x)$ for every $n \in \mathbf{N}$, whence $\omega_-(z) = \omega_-(x)$. Suppose $z \in \mathscr{S}_-$; then also $\omega(z) = \omega_-(x)$ which shows that $p \ll T^m y \ll T^m z \ll q$ for some $p \in \omega_-(x)$, $q \in \omega(x)$ and $m \in \mathbf{Z}_+$. Thus $z \in Y$, an open set in V, which contradicts our choice of $z \in \partial_+ Y \subset \partial Y$. We have proved $\partial_+ Y \subset \mathscr{U}_-$.

(ii) To prove $T(\partial_+ Y) \subset \partial_+ Y$ we fix an arbitrary $z \in \partial_+ Y$. Since Y is open in V, $T(Y) \subset Y$ and $T^{-1}(Y) \subset Y$, from $z \in \partial Y$ and $Tz \in T(\mathrm{Cl}(Y)) \subset \mathrm{Cl}(Y)$ we obtain $Tz \in \partial Y$. Furthermore, we have $y \ll z$ for some $y \in Y$, whence $Ty \ll Tz$ with $Ty \in Y$. We have proved $Tz \in \partial_+ Y$ as desired.

For all $g \in G$, the equality $g \cdot \partial_+ Y = \partial_+ Y$ follows from $g \cdot Y = Y$ in a similar way.

(iii) Suppose $\partial_+ Y$ is not unordered, i.e., $u < v$ for some u, $v \in \partial_+ Y$. Then also $Tu \ll Tv$ with Tu, $Tv \in \partial_+ Y$, and so we may assume $u \ll v$ for some u, $v \in \partial_+ Y$. Similarly as in the proof of Part (i) we can construct a sequence $y_n \in Y$ such that $y_1 \ll y_2 \ll \cdots \ll v$ and $y_n \to v$ as $n \to \infty$. Pick $k \in \mathbf{N}$ so large that $u \ll y_k \ll v$. Also $y \ll u$ for some $y \in Y$. But Y is open and order-convex in V, and therefore $u \in [y, y_k] \subset Y$ contradicts $u \in \partial Y$. We conclude that $\partial_+ Y$ is unordered. ∎

In order to prove that $\partial_+ Y$ is a Lipschitz manifold of codimension in $\hat{V} = (V, \|\cdot\|_e)$, the space V with the order topology, we need the following concepts from [25, Def. 1.0]:

DEFINITION 3.3. A pair (A, B) of subsets A, B of X is called an *order decomposition* of X if it has the following five properties: (i) $A \neq \varnothing$ and $B \neq \varnothing$; (ii) A and B are closed; (iii) A is lower closed and B is upper closed; (iv) $A \cup B = X$; and (v) $\mathrm{Int}(A \cap B) = \varnothing$.

An order decomposition (A, B) of X is called *invariant* if $T(A) \subset A$ and $T(B) \subset B$. The set $H = A \cap B$ (possibly empty) is called the *boundary* of the order decomposition (A, B) of X. A *d-hypersurface* is any nonempty subset H of X such that $H = A \cap B$ for some order decomposition (A, B) of X.

Note that the boundary H of an order decomposition (A, B) of X satisfies $H = \partial A = \partial B$, and H is invariant and unordered whenever (A, B) is invariant. To see that H is unordered suppose $x < y$ for some x, $y \in H$. Then also $Tx \ll Ty$ in $T(H) \subset H$, and therefore we may assume $x \ll y$ in H.

But H is order-convex since A is lower closed and B is upper closed, and consequently $[\![x, y]\!] \subset H$ which contradicts $\operatorname{Int}(H) = \varnothing$.

The following result about invariant d-hypersurfaces was proved in [25, Prop. 1.2]:

PROPOSITION 3.4. *Let V satisfy (V), and let X be a nonempty, open, and order-convex subset of V. If (A, B) is an order decomposition of X, then its boundary $H = A \cap B$ is a Lipschitz manifold of codimension one in $\hat{V} = (V, \|\cdot\|_e)$.*

Now we are ready to prove

PROPOSITION 3.5. *Let all hypotheses of Lemma 3.1 be satisfied. Let $x \in \mathscr{U}_-$, and define the set Y as in Lemma 3.1. Then $\partial_+ Y$ is a Lipschitz manifold of codimension one in $\hat{V} = (V, \|\cdot\|_e)$.*

A corresponding result holds for $x \in \mathscr{U}_+$.

Proof. Set

$$A^0 = \{ y \in X \colon T^m y \ll q \text{ for some } q \in \omega(x) \text{ and } m \in \mathbf{Z}_+ \},$$

and define $A = \operatorname{Cl}(A^0)$ and $B = X \backslash A^0$. Clearly $A^0 \cap \omega(x) = \varnothing$ and $\omega(x) \subset H = A \cap B$, whence $A \neq \varnothing$ and $B \neq \varnothing$. It is easy to check that A^0 is open in V and lower closed in X. Hence ∂A^0 is unordered, and therefore we have $\operatorname{Int}(\partial A^0) = \varnothing$ and $\operatorname{Int}(A) = A^0$. Moreover, A is lower closed and B is upper closed in X, by [25, Lemma 2.1]. Obviously, both A and B are closed in X, and $A \cup B = X$. Since $A \cap B = \partial A^0$, we have also $\operatorname{Int}(A \cap B) = \varnothing$. We conclude that (A, B) is an order decomposition of X. Finally, from $T\omega(x) = \omega(x)$ we obtain $T(A^0) \subset A^0$ and $T^{-1}(A^0) \subset A^0$, whence (A, B) is invariant. We conclude from Proposition 3.4 that $H = A \cap B$ is a Lipschitz manifold of codimension one in $\hat{V} = (V, \|\cdot\|_e)$ with $T(H) \subset H$.

Next we claim $\partial_+ Y \subset H$. It is obvious that $Y \subset A^0$, and so $\partial_+ Y \subset A$. Hence, it suffices to show $\partial_+ Y \cap A^0 = \varnothing$. On the contrary, suppose there is $z \in \partial_+ Y \cap A^0 \neq \varnothing$. Then $y \ll z$ for some $y \in Y$ which means that there exist $p \in \omega_-(x)$, $q \in \omega(x)$, and $m \in \mathbf{Z}_+$ such that $p \ll T^m y \ll T^m z \ll q$. Hence, also $z \in Y$ which contradicts $z \in \partial_+ Y$. We have proved $\partial_+ Y \subset H$.

In order to prove that $\partial_+ Y$ is a Lipschitz manifold of codimension one in $\hat{V} = (V, \|\cdot\|_e)$ we only need to show that $\partial_+ Y$ is an open subset of $\hat{H} = (H, \|\cdot\|_e)$, the set H with the order topology. We fix an arbitrary $z \in \partial_+ Y$. So $y \ll z \ll y'$ for some $y \in Y \subset X$ and $y' \in X$. Also $[y, y'] = [y, y']_V \subset X$ since X is order-convex in V. We claim $[\![y, y']\!] \cap H \subset \partial_+ Y$. Indeed, pick any $v \in [\![y, y']\!] \cap H$. From Proposition 1.4 and $T(H) \subset H$ we obtain $\omega_-(x) = \omega(y) \lll \omega(v) \subset H$. Hence $p \ll T^m v$ for some $p \in \omega_-(x)$

and $m \in \mathbf{Z}_+$. Since also $v \in H = \partial A^0$ and A^0 is lower closed, we can construct a sequence $y_n \in A^0$ such that $y \ll y_1 \ll y_2 \ll \cdots \ll v$ and $y_n \to v$ as $n \to \infty$. Pick $k \in \mathbf{N}$ so large that $p \ll T^m y_n \ll T^m v$ for all $n \geq k$. From $y_n \in A^0$ we have $T^{m_n} y_n \ll q_n$ for some $q_n \in \omega(x)$ and $m_n \in \mathbf{Z}_+$. Consequently $y_n \in Y$ for all $n \geq k$ which entails $v \in \partial_+ Y$ as desired. We have shown that $\partial_+ Y$ is open in $\hat{H}$, and thus, we have finished our proof. ∎

Proof of Theorem 0.3(b). It follows from Lemma 3.2(i) and Proposition 3.5 that both $\mathscr{U}_-$ and $\mathscr{U}_+$ are unions of Lipschitz manifolds of codimension one in $\hat{V}$. By Lemma 3.2(ii), both $\mathscr{U}_-$ and $\mathscr{U}_+$ are invariant under T and the action of G. We will show that both these unions are at most countable. Obviously $\mathscr{U}_2 \subset \mathscr{U}_{1/2} = \mathscr{U}_- \cup \mathscr{U}_+$.

Given any $x \in \mathscr{U}_-$, define the set $Y \equiv Y(x)$ as in Lemma 3.1. We claim that, for any pair x, $x' \in \mathscr{U}_-$, we have either $Y(x) = Y(x')$ or else $Y(x) \cap Y(x') = \varnothing$. Indeed, choose any x, $x' \in \mathscr{U}_-$ and suppose there is $y \in Y(x) \cap Y(x') \neq \varnothing$. Hence $p \ll T^m y \ll q$ and $p' \ll T^{m'} y \ll q'$ for some $p \in \omega_-(x) = \omega(y)$, $q \in \omega(x)$, and $m \in \mathbf{Z}_+$, and for some $p' \in \omega_-(x') = \omega(y)$, $q' \in \omega(x')$, and $m' \in \mathbf{Z}_+$. We arrive at $\omega_-(x) = \omega_-(x')$. Thus, if $y' \in Y(x')$, then $p \ll T^m y'$ for some $p \in \omega_-(x)$ and $m \in \mathbf{Z}_+$, and also $\omega(y') = \omega_-(x) \lll \omega(x)$ whence $T^k y' \ll q$ for some $q \in \omega(x)$ and $k \in \mathbf{Z}_+$. Consequently $y' \in Y(x)$, and so we have proved $Y(x') \subset Y(x)$. Analogously $Y(x) \subset Y(x')$, and therefore $Y(x) = Y(x')$ as claimed.

Since V is separable and each $Y(x)$ is nonempty and open in V, the collection $\{Y(x) : x \in \mathscr{U}_-\}$ contains at most countably many distinct elements. We conclude that $\mathscr{U}_- = \cup \{\partial_+ Y(x) : x \in \mathscr{U}_-\}$ is a union of at most countably many Lipschitz manifolds of codimension one in $\hat{V}$. Analogously, $\mathscr{U}_+$ is such a union.

Finally, consider any Gaussian measure μ on V. For every $x \in \mathscr{U}_-$, the set $\partial_+ Y(x)$ is unordered by Lemma 3.2(iii), and Borel in V by Proposition 3.5. Consequently, we may apply a result of Hirsch [14, Lemma 7.7(a)] to conclude that $\mu(\partial_+ Y(x)) = 0$. The countability of the union $\mathscr{U}_- = \cup \{\partial_+ Y(x) : x \in \mathscr{U}_-\}$ then implies $\mu(\mathscr{U}_-) = 0$. Similarly $\mu(\mathscr{U}_+) = 0$. We arrive at $\mu(\mathscr{U}_2) = 0$ as desired. Our proof of Theorem 0.3(b) is now complete. ∎

The reader is referred to [16] for general facts about Gaussian measures in Banach spaces, and to [4, 23] for descriptions of their null sets. Some additional details about null sets can be found in [14, Lemma 7.7].

Proof of Theorem 0.1(b). This result follows directly from Theorem 0.3(b). Note that $\hat{V} = V$ for $V = C_{\Omega\text{-per}}(\mathbf{R}^N \to \mathbf{R}^n)$. ∎

4. EXAMPLES

In this section we present a few more applications of Theorem 0.3 to problems similar to the IVP (1) in the Introduction.

EXAMPLE 4.1. Consider the following reaction–diffusion-type initial-boundary value problem (IBVP) for an unknown vector-valued function $u(t, x) \in \mathbf{R}^n$ of the time–space variable $(t, x) \in \mathbf{R}^1_+ \times \Omega$:

$$\frac{\partial u}{\partial t} = D(t, x)\, \Delta u + F(t, x, u), \qquad t > 0,\, x \in \Omega; \tag{7}$$

$$(\mathbf{B}u)(t, x) = 0, \qquad (t, x) \in \mathbf{R}^1_+ \times \Omega; \tag{7_b}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega. \tag{7_i}$$

Here Ω is an open bounded domain in $\mathbf{R}^N$ with a $C^{1+\mu}$-boundary $\partial\Omega$, for some $\mu \in (0, 1)$. We assume also that $D \in C^{\mu/2, \mu}(\mathbf{R}^1_+ \times \bar{\Omega} \to \mathbf{R}^{n \times n})$ is a diagonal matrix with positive entries, and all mappings F and $\partial F_j/\partial u_k$ a re continuous in $\mathbf{R}^1_+ \times \bar{\Omega} \times \mathbf{R}^n$ with $F_j(\cdot, \cdot, u)$, $(\partial F_j/\partial u_k)(\cdot, \cdot, u) \in C^{\mu/2, \mu}(\mathbf{R}^1_+ \times \bar{\Omega} \to \mathbf{R}^1)$ uniformly for u in compact subsets of $\mathbf{R}^n$. Both $D(t, x)$ and $F(t, x, \cdot)$ are assumed to be *τ-periodic* in time t with a period $\tau > 0$. We consider a regular linear boundary operator $\mathbf{B}$ on $\partial\Omega$ of either Neumann or oblique derivative type. For simplicity, $\mathbf{B}$ is assumed to be time-independent. We take $V = C(\bar{\Omega} \to \mathbf{R}^n)$.

This time we do not have any obvious system of ODEs attracting every stable solution of our reaction–diffusion problem. Let $T\colon V \to V$ be the Poincaré map for this problem. We assume that every $f \in V$ has a bounded orbit $\mathcal{O}^+(f)$; hence $\mathcal{O}^+(f)$ is also relatively compact in V, since T maps bounded sets into relatively compact sets, cf. [2] or [17]. In particular, T is strongly increasing and ω-compact in every closed order interval $[a, b]$ in V. All remaining hypotheses are the same as in the IVP in the Introduction. Similarly to [21, Sect. 3] we assume that G is a compact connected subgroup of $SO(N)$, the group of all orthogonal order-preserving linear transformations of $\mathbf{R}^N$ onto itself, whose action leaves Ω invariant, i.e., $g \cdot \Omega = \Omega$ for all $g \in G$. The action of G on V is defined by $g \cdot f = f \circ g$, i.e.,

$$\gamma(g, f)(x) = f(g \cdot x) \qquad \text{for all } g \in G,\ f \in V \text{ and } x \in \bar{\Omega}.$$

Finally, we assume that both $D(t, x)$ and $F(t, x, u)$ are spatially symmetric under G, i.e., $D(t, g \cdot x) = D(t, x)$ and $F(t, g \cdot x, u) = F(t, x, u)$ for all $g \in G$ and $(t, x, u) \in \mathbf{R}^1_+ \times \Omega \times \mathbf{R}^n$. Then γ commutes with T by [14, Proof of Thm. 5.4].

We denote by $V_{\#}$ the closed linear subspace of V of all symmetric functions $f \in V$, i.e., $g \cdot f$ for all $g \in G$. Applying Theorem 0.3 we conclude that, in order to determine the asymptotic behavior of $T^n f$ as $n \to \infty$ in $\mathbf{Z}_+$, for a given $f \in \mathscr{S}_{1/2}$ in V, it suffices to investigate the dynamics of T on the totally invariant compact set $\omega(f) \subset V_\#$. Hence, we may restrict ourselves to the restriction $T_\# = T_{|V_\#}$ of T to $V_\#$. Especially, if G is "sufficiently rich," this means a considerable reduction in computing time whenever one tries to compute the dynamics of T on a local attractor $K \subset V$, since $K \subset [a, b]$ for some $a, b \in V_\#$ with $a \ll b$, and so $K \cap \mathscr{S}_{1/2}$ is pointwise attracted by $K' = \cup\{\omega(x): x \in [a, b] \cap V_\#\}$ which is a relatively compact, totally invariant subset of $V_\#$. This reduction is apparent from the following two examples.

EXAMPLE 4.2. Take $G = SO(N)$ in Example 4.1, i.e., consider the IBVP (7) in an open ball $\Omega = B_R \equiv \{x \in \mathbf{R}^N: |x| < R\}$ with radius $R \in (0, \infty)$, and with radially symmetric functions $D(t, x) = D(t, r)$ and $F(t, x, u) = F(t, r, u)$, where $r = |x| \in [0, R]$. For simplicity, assume the Neumann boundary conditions on ∂B_R, i.e., $\partial u/\partial r = 0$ for $r = R$. Then, given any $f \in \mathscr{S}_{1/2}$ in V, the dynamics of T on $\omega(f)$ is determined by the following radially symmetric system in $V_\# = C([0, R] \to \mathbf{R}^n)$:

$$\frac{\partial u}{\partial t} = D(t, r)\left(\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r}\right) + F(t, r, u), \qquad t > 0,\ 0 \leqslant r < R; \tag{8}$$

$$\frac{\partial u}{\partial r} = 0, \qquad t > 0,\ r = 0, \text{ and } R; \tag{8_{b}}$$

$$u(0, r) = u_0(r), \qquad 0 \leqslant r \leqslant R. \tag{8_{i}}$$

This system in one spatial dimension allows considerably faster computations than the original system in B_R. Here a simple finite difference method usually suffices for computer simulations of asymptotically stable orbits.

EXAMPLE 4.3. We consider the IBVP (7) in an open solid torus $\Omega = S^1_\rho \times D^2_R \subset \mathbf{R}^3$, where $\rho \in (0, \infty)$ is the radius of the central circle S^1_ρ and $R \in (0, \rho)$ is the radius of the open disk D^2_R which is the cross-section of our solid torus perpendicular to the central circle. Since the Laplacian Δ is invariant under every orthogonal (distance-preserving) linear transformation of coordinates, we may choose the coordinates $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ such that a generic point $x \in \Omega$ has the coordinates

$$x_1 = (\rho + r\cos\varphi)\cos\theta$$

$$x_2 = (\rho + r\cos\varphi)\sin\theta$$

$$x_3 = r\sin\varphi,$$

where (θ, r, φ) are the new variables satisfying θ, $\varphi \in (-\pi, \pi]$ and $r \in [0, R)$. Observe that r is the distance from x to the central circle. Assume that both $D(t, x)$ and $F(t, x, u)$ are radially symmetric relative to S^1_ρ, i.e., $D(t, x) = D(t, r, \varphi)$ and $F(t, x, u) = F(t, r, \varphi, u)$, where $r \in [0, R]$ and $\varphi \in (-\pi, \pi]$. Again, for simplicity, assume the Neumann boundary conditions on $\partial(S^1_\rho \times D^2_R) \simeq S^1_\rho \times S^1_R$ which is a torus, i.e., $\partial u/\partial \tau = 0$ for $r = R$. Then, given any $f \in \mathscr{S}_{1/2}$ in V, the dynamics of T on $\omega(f)$ is determined by a system in $V_\# = C(D^2_R \to \mathbf{R}^n)$ which depends spatially only on $(r, \varphi) \in [0, R] \times (-\pi, \pi]$ and is very similar to (8), (8_b), and (8_i). We leave the details to the reader.

Our last example shows that the connectedness of the symmetry group G in Theorem 0.3 is essential:

Example 4.4. Consider the following system of n difference equations in $V = \mathbf{R}^n$; $n \geqslant 2$,

$$u_i' = \beta \sum_{j=1}^{n} u_j + \alpha u_{i+1}(1 + |u_{i+1}|)^{-1}, \qquad i = 1, ..., n,$$

where $u = (u_1, ..., u_n) \in \mathbf{R}^n$ with $u_{n+1} \equiv u_1$. Define $T: \mathbf{R}^n \to \mathbf{R}^n$ by $Tu = u'$. Here $\alpha \in (1, \infty)$ and $\beta \in (0, \infty)$ are constants, and hence, T is strongly increasing. We fix the constant α and vary β near 0; we write $T \equiv T_\beta$. Take $\beta \in (0, 1/n)$; then the compact order interval $[-e, e] \subset \mathbf{R}^n$, where $e = (\xi, ..., \xi)$ with $\xi \in (0, \infty)$ defined by $1 = n\beta + \alpha(1 + \xi)^{-1}$, attracts every point $u \in \mathbf{R}^n$, i.e. $\omega(u) \subset [-e, e]$, by monotonicity combined with $Te = e < Tv < v$ for all $v = (\eta, ..., \eta) \in \mathbf{R}^n$ with $\eta > \xi$. Observe that T is odd $(T(-u) = -Tu$ for $u \in \mathbf{R}^n)$. Next we show that T has an asymptotically stable n-cycle in $[[-e, e]]$ of the form $C = \{p, g \cdot p, g^2 \cdot p, ..., g^{n-1} \cdot p\}$, where

$$p = (a, -b, -b, ..., -b) \qquad \text{for some } a, b \in (0, \infty),$$

and g is the cyclic permutation

$$g(i) = i + 1 \qquad \text{for } i = 1, ..., n-1, \qquad \text{and} \qquad g(n) = 1,$$

acting on $\mathbf{R}^n$ by $g \cdot (u_1, ..., u_n) = (u_{g(1)}, ..., u_{g(n)})$.

First observe that g generates a cyclic group G of order n acting on $\mathbf{R}^n$ whose action commutes with T. Hence, a and b satisfy the two equations

$$a = \beta[a - (n-1)\,b] + \alpha a(1 + a)^{-1},$$

$$-b = \beta[a - (n-1)\,b] - \alpha b(1 + b)^{-1}.$$

Let $(a, b) \equiv (a_\beta, b_\beta)$ depending upon β. If $\beta = 0$, we obtain $a_0 = b_0 = \alpha - 1 > 0$ with T_0^n is a contraction near the cycle $C \equiv C_0$, since the spectrum of $DT_0(T_0^k p)$, the Jacobian matrix of T_0 at $T_0^k p \in C$ which is independent from $k = 0, ..., n-1$, consists of the eigenvalues $\alpha^{-1} e^{2il\pi/n} \in \mathbf{C}$, $l = 0, ..., n-1$. Elementary local bifurcation and perturbation theories for $\beta \in \mathbf{C}$ near 0, cf. [15], guarantee the existence of a constant $\beta' \in (0, \infty)$ with the following property: The system for (a, b) has a solution (a_β, b_β) depending analytically upon $\beta \in [0, \beta')$ and such that T_β^n is a contraction near the cycle $C \equiv C_\beta$. In particular, C is asymptotically stable.

5. Concluding Remarks

Remark 1. There is an essential difference between the applicability of spatial symmetry to strongly increasing discrete-time semigroups (or time-periodic processes) and continuous-time semiflows. In this article we have used spatial symmetry in order to investigate convergence of stable trajectories in discrete-time semigroups (cf. Theorem 0.1 and Corollary 0.2), whereas Mierczyński and Poláčik [21] already knew convergence (or quasiconvergence) from Hirsch [11, 12, 14] and investigated only spatial symmetry of the sets of equilibria attracting stable trajectories in continuous-time semiflows. Simply, in this article we have substituted the connectedness of the time semigroup $\mathbf{R}_+^1 = [0, \infty)$ by that of the group G.

Remark 2. An example similar to our Example 4.4 has been constructed independently by Dancer and Hess [6, Sect. 2] in connection with the Limit Set Dichotomy for strongly increasing discrete-time semigroups. Both examples show the existence of asymptotically stable cycles of order >1. A third such example, a time-periodic positive feedback system of four ODEs, has been constructed in [27, Sect. 4] with only numerical evidence of asymptotically stable 8-cycles for the corresponding Poincaré map. This fact is in strong contrast with the results of [14, Thm. 6.8] for strongly increasing continuous-time semiflows, and [9, Thm. 4.2] for time-periodic cooperative systems of two ODEs.

Remark 3. The connectedness of G is essential even for semiflows treated in [21]. Namely, Matano and Mimura [20, Thm. A] showed that, given a very general time and space independent competition–diffusion system for two species, one can always find a bounded C^2-domain $\Omega \subset \mathbf{R}^2$ symmetric with respect to the vertical axis x_2, and such that the given system in Ω with the Neumann boundary conditions on $\partial\Omega$ has a non-symmetric asymptotically stable equilibrium. In this case $G = \{e, g\}$, where $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $g(x_1, x_2) = (-x_1, x_2)$.

Remark 4. On the other hand, assuming large diffusivity, but no cooperativeness or symmetry for the domain, Hale [7] (cf. also [8, Sect. 4.10]) showed that the dynamics on a local (or global) attractor for a system of time and space independent reaction–diffusion equations in a smooth bounded domain $\Omega \subset \mathbf{R}^N$ with the Neumann boundary conditions on $\partial\Omega$ coincides with the dynamics on this attractor for the corresponding system of ODEs, see our Introduction.

Remark 5. Time-periodic reaction–diffusion equations with rather special nonlinearities were studied also in [1, 6, 10, 25], where convergence of every trajectory to a time-periodic solution, with the period of the equation, was established by various methods.

Remark 6. A local (or global) attractor for a general discrete-time dynamical system (cf. [8, Sect. 2.4] for a definition) is *always* invariant under the action of any connected metrizable group G whose action commutes with the dynamical system. More precisely, we have the following result:

PROPOSITION 5.1. *Let X be a metrizable topological space, $T: X \to X$ continuous, and let G be a connected metrizable group whose action on X commutes with T. Assume that K is a nonempty compact subset of X, $T(K) = K$, and there exists an open subset U of X containing K and satisfying the following condition: Given any open subset W of X containing K, there exists $n \equiv n(W) \in \mathbf{Z}_+$ such that $T^k(U) \subset W$ for all integers $k \geqslant n$.*
Then $g \cdot K = K$ for all $g \in G$.

Proof. Let $G_0 = \{g \in G: g \cdot K \subset K\}$. Obviously $e \in G_0$. Since K is compact, and hence, closed in X, also G_0 is closed in G. We claim that G_0 is also open in G. Indeed, pick any $g_0 \in G_0$. Since K is compact, there exists an open neighborhood Γ of g_0 in G such that $\Gamma \cdot K \subset U$. Let W be any open subset of X containing K. Then for all $g \in \Gamma$ and $k \geqslant n(W)$ we have

$$g \cdot K = g \cdot T^k(K) = T^k(g \cdot K) \subset T^k(U) \subset W.$$

But W is an arbitrary open subset of X containing K, and so we arrive at $g \cdot K \subset K$ for all $g \in \Gamma$, i.e., $\Gamma \subset G_0$. Hence G_0 is also open in G. Since G is connected, we must have $G_0 = G$ which means $G \cdot K \subset K$. Then $g \cdot K = K$ for all $g \in G$ follows from g, g^{-1}, $e \in G$.

ACKNOWLEDGMENTS

The author expresses his gratitude to Professor Hiroshi Matano (University of Tokyo) for a number of valuable comments on an earlier version of this paper.

REFERENCES

1. N. D. ALIKAKOS, P. HESS, AND H. MATANO, Discrete order preserving semigroups and stability for periodic parabolic differential equations, *J. Differential Equations* **82** (1989), 322–341.
2. H. AMANN, Existence and regularity for semilinear parabolic evolution equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **11** (1984), 593–676.
3. H. AMANN, Global existence for semilinear parabolic systems, *J. Reine Angew. Math.* **360** (1985), 47–83.
4. N. ARONSZAJN, Differentiability of Lipschitzian mappings between Banach spaces, *Studia Math.* **57** (1976), 147–190.
5. X-Y. CHEN AND H. MATANO, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, *J. Differential Equations* **78** (1989), 160–190.
6. E. N. DANCER AND P. HESS, Stability of fixed points for order-preserving discrete-time dynamical systems, *J. Reine Angew. Math.* **419** (1991), 125–139.
7. J. K. HALE, Large diffusivity and asymptotic behavior in parabolic systems, *J. Math. Anal. Appl.* **118** (1986), 455–466.
8. J. K. HALE, Asymptotic behavior of dissipative systems, *in* "Math. Surveys and Monographs," Vol. 25, Amer. Math. Soc., Providence, RI, 1988.
9. J. K. HALE AND A. S. SOMOLINOS, Competition for fluctuating nutrient, *J. Math. Biol.* **18** (1983), 255–280.
10. P. HESS AND H. F. WEINBERGER, Convergence to spatial–temporal clines in the Fisher equation with time-periodic fitnesses, *J. Math. Biol.* **28** (1990), 83–98.
11. M. W. HIRSCH, Differential equations and convergence almost everywhere in strongly monotone semiflows, *Contemp. Math.* **17** (1983), 267–285.
12. M. W. HIRSCH, The dynamical systems approach to differential equations, *Bull. Amer. Math. Soc.* **11** (1984), 1–64.
13. M. W. HIRSCH, Attractors for discrete-time monotone dynamical systems in strongly ordered spaces, *in* "Geometry and Topology," Lecture Notes in Math., Vol. 1167, pp. 141–153, Springer-Verlag, Berlin/Heidelberg/New York, 1985.
14. M. W. HIRSCH, Stability and convergence in strongly monotone dynamical systems, *J. Reine Angew. Math.* **383** (1988), 1–53.
15. G. IOOSS AND D. D. JOSEPH, "Elementary Stability and Bifurcation Theory," Springer-Verlag, New York/Heidelberg/Berlin, 1980.
16. H-H. KUO, Gaussian measures in banach spaces, *in* "Lecture Notes in Math.," Vol. 463, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
17. O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URALTSEVA, Linear and quasilinear equations of parabolic type, *in* "Translations of Math. Monographs," Vol. 23, Amer. Math. Soc., Providence, RI, 1968.
18. H. MATANO, Asymptotic behaviour and stability of solutions of semilinear diffusion equations, *Publ. Res. Inst. Math. Sci.* **15** (1979), 401–454.
19. H. MATANO, Existence of nontrivial unstable sets of equilibriums of strongly order-preserving systems, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30** (1983), 645–673.
20. H. MATANO AND M. MIMURA, Pattern formation in competition–diffusion systems in nonconvex domains, *Publ. Res. Inst. Math. Sci.* **19** (1983), 1049–1079.
21. J. MIERCZYŃSKI AND P. POLÁČIK, Group actions on strongly monotone dynamical systems, *Math. Ann.* **283** (1989), 1–11.
22. D. MONTGOMERY AND L. ZIPPIN, Topological transformation groups, *in* "Interscience Tracts in Pure and Applied Mathematics," Vol. 1, Interscience, New York, 1955.
23. R. R. PHELPS, Gaussian null sets and differentiability of Lipschitz map on Banach spaces, *Pacific J. Math.* **77**, No. 2 (1978), 523–531.

24. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1967.
25. P. TAKÁČ, Convergence to equilibrium on invariant d-hypersurfaces for strongly increasing discrete-time semigroups, *J. Math. Anal. Appl.* **148** (1990), 223–244.
26. P. TAKÁČ, Domains of attraction of generic ω-limit sets for strongly monotone discrete-time semigroups, *J. Reine Angew. Math.* **423** (1992), 101–173.
27. P. TAKÁČ, Linearly stable subharmonic orbits in strongly monotone time-periodic dynamical systems, *Proc. Amer. Math. Soc.* **115** (1992), 691–698.